

RESEARCH STATEMENT

DAVID WEN

INTRODUCTION

My research is in Algebraic Geometry, specifically in birational geometry which is the study of algebraic varieties up to birational equivalence i.e., having isomorphic function fields. One of the main research programs in Algebraic Geometry is the classification of all varieties and towards this goal two approaches developed. One being moduli spaces, that classifies up to isomorphisms, and the other is birational geometry that classifies up to birational maps. My research focuses on the latter which allows a degree of flexibility because we can modify our varieties with birational modifications into “nicer” models which we can work with and classify.

The first big results towards birational classification were formulated by the Italian school of algebraic geometry with the likes of Castelnuovo, Enriques, Severi and more. They showed that a birational map between algebraic surfaces decomposed into simpler birational maps which eventually led to a birational classification of complex algebraic surfaces. This classification was achieved through a procedure consisting of a sequence of contractions that would terminate, thus leaving a distinguished smooth surface birational to the original which we called a minimal model. It was these minimal models of surfaces that were classified into the Kodaira-Enriques classification.

The procedure of contractions became the framework of the minimal model program with the goal of obtaining minimal models of higher dimensional algebraic varieties. Work on the threefold case began in the 1980s and much had to be generalized. The definition of a minimal model was expanded upon and allowed to have singularities. Additionally, more types of birational maps had to be accounted for and understood. By the late 80s, with efforts by Mori and many others, the threefold case was resolved and we now know that smooth projective threefolds have a minimal model or degenerates to a Mori fiber space.

The success of the threefold case was promising but the techniques needed to be expanded in higher dimensions. The more modern development of the higher dimensional minimal model program approached the problem through cohomology and by way of induction as a means to obtain all dimensions. This gave way for the existence of minimal models of smooth projective fourfolds and many more varieties (like general type) in higher dimensions.

This setting of higher dimensional birational geometry is where I work. My research involves studying and analyzing the properties and structure of minimal models of certain classes of varieties. Currently, I am investigating minimal models of elliptic fibrations.

BACKGROUND

A variety will be a normal $\mathbb{Q}$ -factorial complex projective variety. A log pair, (X, Δ) , is a variety X with a Weil divisor Δ , with coefficients in $\mathbb{Q} \cap [0, 1]$. Associated to X is a divisor called the canonical divisor, denoted K_X , which corresponds to the line bundle of holomorphic top form. In the case of a log pair (X, Δ) , we have an analog called the log canonical divisor which is $K_X + \Delta$.

Definition. Given a divisor D on X , we say that D is nef if for every curve, $C \subset X$ we have that $D \cdot C \geq 0$, where $D \cdot C$ is the intersection number of D and C . In the case where canonical divisor K_X is nef, we call X a minimal model. Analogously, we call (X, Δ) a log minimal model if $K_X + \Delta$ is nef.

The definition above hints at the process of the minimal model program, which is to contract all curves that negatively intersect with K_X . What's left after these contractions, if this process terminates, would be a minimal model. My interest is the relation between minimal models and elliptic fibrations.

Definition. An elliptic fibration is a variety X with a morphism $\pi : X \rightarrow B$ such that a general fiber of π is an elliptic curve. We say that $\pi : X \rightarrow B$ is an elliptic fibration with section if there is a morphism $s : B \rightarrow X$ such that $\pi \circ s = \text{id}_B$.

A rational elliptic fibration is a rational map $\pi : X \dashrightarrow B$ with an open dense set $U \subset B$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$ is an elliptic fibration.

The two dimensional case of elliptic surface was studied by Kodaira. Minimal models of elliptic surfaces are achieved by contracting (-1) -curves that are contained within the fibers, resulting in a minimal model that is still an elliptic surface. In dimension 3, there are more technicalities but we have the following result from Grassi while higher dimensions (≥ 4) is open.

Theorem ([4, Thm 1.1]). Let $X_0 \rightarrow S_0$ be an elliptic threefold which is not uniruled. Then there exists a birationally equivalent fibration $\bar{\pi} : \bar{X} \rightarrow \bar{S}$, such that $\bar{X}$ has at worst terminal and $\bar{S}$ log terminal singularities. Furthermore $K_{\bar{X}}$ is nef and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{S}} + \bar{\Lambda})$, where $\bar{\Lambda}$ is a $\mathbb{Q}$ -boundary divisor. Thus the canonical bundle is a pullback of a $\mathbb{Q}$ -bundle on $\bar{S}$.

In Grassi's result, $\bar{X}$ is a minimal model that is also an elliptic fibration, which seems unusual because in general a contraction wouldn't preserve a fibration structure. Part of the proof of Grassi's theorem utilizes a canonical bundle formula for elliptic fibrations from [3] given below:

$$mK_X = \pi^* \left(mK_B + m\pi_*(K_{X/B}) + m \sum \left(\frac{m_i - 1}{m_i} Y_i \right) \right) + mE - mG \quad (1)$$

Where $\pi_*(K_{X/B})$ means the divisors corresponding $\pi_*\mathcal{O}_B(K_{X/B})$. With enough hypotheses and letting $\Lambda = \pi_*(K_{X/S}) + \sum \frac{m_i - 1}{m_i} Y_i$, we have (S, Λ) is a log pair of which it is possible to run the log minimal model program. Running the log minimal model program on (S, Λ) gives the log pair $(\bar{S}, \bar{\Lambda})$ in the theorem above.

To maintain the fiber structure, there needs to be some control on how the minimal model program runs on the base space and on the total space. This is achieved by a generalized Zariski Decomposition called the Fujita-Zariski Decomposition.

Definition ([1, Def 1.1], cf. [3, Def 1.18]). Let D be an $\mathbb{R}$ -Cartier divisor on X , a normal variety. A Fujita-Zariski Decomposition of D is an expression $D = P + N$ such that P and N are $\mathbb{R}$ -Cartier with P is nef and $N \geq 0$. Lastly, if $f : W \rightarrow X$ is a projective birational morphism from a normal variety and $f^*(D) = P' + N'$ with P' nef and N' effective, then $P' \leq f^*(P)$.

This is not the original definition from Fujita in [3] but it is an equivalent generalized definition from Birkar given in [1]. Grassi used this decomposition to explicitly show that the appropriate divisors are contracted when running the relative minimal model program, thus resulting in the equation $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{S}} + \bar{\Lambda})$.

RESULTS

Generalizing towards higher dimensions is difficult partially because there are psuedo-effective divisors in higher dimensions that may not have a Fujita-Zarsiki decompositions. But working birationally we have the following result from Birkar:

Theorem ([1, Thm 1.5]). *Assume the log minimal model program for $\mathbb{Q}$ -factorial divisorial log terminal pairs in dimension $n - 1$. Let (X, Δ) be log canonical of dimension n , then $K_X + \Delta$ birationally has a Fujita-Zariski Decomposition if and only if (X, Δ) has a log minimal model.*

For an elliptic fourfold, $\pi : X \rightarrow B$, we have that B is a threefold base with an associated divisor Δ coming from the canonical bundle formula making (B, Δ) log terminal. The case of the log minimal model program for log canonical threefolds have been established thus for the log terminal pair (B, Δ) , we have that $K_B + \Delta$ birationally has a Fujita-Zariski decomposition. This means there is a sequence of blow ups $g : \tilde{B} \rightarrow B$ such that $g^*(K_B + \Delta)$ has a Fujita-Zariski decomposition. This now brings into question how does this affect the canonical bundle formula of X and it's relation to $g^*(K_B + \Delta)$.

In [5], an analysis partially addressing this question was done for Weierstrass models, which is an elliptic fibration that locally looks like an affine Weierstrass equation. We can pull back a Weierstrass model on B to a Weierstrass model on $\tilde{B}$ and so it is possible to compare the log canonical divisors of (B, Δ) to $(\tilde{B}, \tilde{\Delta})$ in relation to the canonical bundle formula. These were the tools I used to show that with enough reasonable hypotheses on a Weierstrass model, $\pi : X \rightarrow B$, K_X birationally has a Fujita-Zariski decomposition.

With K_X birationally having a Fujita-Zariski decomposition, I showed that that running the minimal model program would contract the appropriate divisors in this decomposition thus getting rid of the “negative” terms in the canonical bundle formula. The whole process gives a commutative diagram:

$$\begin{array}{ccccc}
 & & \tilde{X} & & \\
 & \swarrow \tilde{g} & & \searrow \mu & \\
 X & & & & \bar{X} \\
 \downarrow \pi & & \downarrow \tilde{\pi} & \searrow \epsilon & \downarrow \bar{\pi} \\
 & & (\tilde{B}, \tilde{\Delta}) & & \\
 & \swarrow g & & \searrow h & \\
 (B, \Delta) & & & & (\bar{B}, \bar{\Delta}) \\
 & \dashrightarrow \psi & & &
 \end{array}$$

Where we have that following:

- X is a Weierstrass model over B and Δ is the divisor associated to $\pi_*\mathcal{O}_X(K_{X/B})$
- $(\bar{B}, \bar{\Delta})$ is the log minimal model of (B, Δ) and $\tilde{B}$ is a common log resolution where $g^*(K_B + \Delta)$ has a Fujita-Zariski Decomposition.
- $\tilde{X}$ is obtained by taking the fiber product of X and $\tilde{B}$ and then resolving the singularities. So $\tilde{\pi} : \tilde{X} \rightarrow \tilde{B}$ is a elliptic fibration between smooth projective varieties.
- $\bar{X}$ is obtained by running the minimal model program on $\tilde{X}$. These contractions result in the equation $K_{\bar{X}} \equiv \tilde{\pi}^*(K_{\tilde{B}} + \bar{\Delta})$.

We have that the exceptional divisors of μ are exactly the divisors over the exceptional locus of h . This shows that outside of this exceptional set, the fibration structure is preserved by μ . The result is the following theorems and corollaries:

Theorem 1. *Let $\pi : X \rightarrow B$ be a Weierstrass model, Δ the divisor associated $\pi_*\mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a log terminal threefold with a log minimal model $(\bar{B}, \bar{\Delta})$. Then there exists a birationally equivalent rational elliptic fibration $\bar{\pi} : \bar{X} \dashrightarrow \bar{B}$, such that $\bar{X}$ is a minimal model of X and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.*

Corollary 2. *With the assumptions above, $\bar{\pi}$ is defined in codimension 1 and there exists a non-empty open $U \subset \bar{B}$ such that $\text{codim}(\bar{B} \setminus U) \geq 2$ and $\bar{\pi}$ is a flat elliptic fibration over U .*

Corollary 3. *With the assumptions above, any minimal model of X with at worst terminal singularities is a rational elliptic fibration over $\bar{B}$*

Theorem 4. *With the assumptions above, the canonical model of $\bar{X}$ is isomorphic to the log canonical model of $(\bar{B}, \bar{\Delta})$. Equivalently, the canonical ring of $\bar{X}$ is isomorphic to the log canonical ring of $(\bar{B}, \bar{\Delta})$.*

FUTURE DIRECTIONS

Currently, I am further investigating the birational geometry of Weierstrass models. The above theorem establishes a relation between minimal models of a Weierstrass model and it's base but how the minimal model program is affecting the fibration structure is still unknown. A few question I want to answer is: Is it possible to realize the map $\bar{\pi}$ as a morphism instead of a rational map? How does the birational properties of a Weierstrass Model interact with the minimal model program? What is the behavior of an elliptic fibration after a flip or a flop on the base? Are there conditions to obtain a minimal model that is equidimensional over the base?

Elliptic fibrations with section are very relevant in Physics, especially in F-Theory with elliptically fibered Calabi-Yau varieties. Some recent work in the birational geometry of low dimensional elliptically fibered Calabi-Yau varieties can be seen in [2]. Lastly, while the case with section is very interesting, it is also worthwhile to consider the case of elliptic fibrations without rational section (i.e., is not birational to a Weierstrass model) as these involve fibers with multiplicities and would require a more intrinsic understanding of the fiber structure.

Moving forward, I want to study the birational geometry and minimal models of algebraic fiber spaces, specifically with general fibers having Kodaira dimension 0. Fibrations with

general fibers having trivial canonical divisors, like elliptic fibrations, are a special case of this. These spaces are relevant to the minimal model program through the Iitaka fibration over canonical models which is one of the outputs of the minimal model program (the others being Mori fiber spaces and general type varieties). These algebraic fiber spaces factors into the Iitaka fibration setting up a relation between the canonical models of the total space with the base space and potentially relating minimal models of the total space with that of the base space. Grassi's theorem for elliptic threefolds and my partial generalization for elliptic fourfolds are evidence of such a relation.

Questions I want to answer towards this direction are: Is it possible to generalize these results to higher dimensions and higher dimensional fibers? For instance, does such a theorem hold for $K3$ -fibrations over surfaces? Can we further understand the canonical divisor of these algebraic fiber spaces, possibly through their singular fibers and other variants of the Zariski decompositions? Can we utilize these techniques to obtain minimal models of algebraic fiber spaces in higher dimensions?

References

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